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FORMAL SECTIONS AND DE RHAM COHOMOLOGY OF SEMISTABLE ABELIAN VARIETIES

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ABSTRACT

We give a geometric description of the unit root splitting of the Hodge filtration of the first de Rham cohomology of an ordinary Abelian variety over a local field, as the splitting determined by a formal completion of the universal vectorial extension of the Abelian variety.

1. Introduction

The main purpose of this paper is to give a geometric description of the unit root splitting of the Hodge filtration of the first de Rham cohomology of an ordinary semistable Abelian variety over a local field. More precisely, let K be a finite extension of $\mathbf{Q}_p$, R its ring of integers, m_R its maximal ideal and k the residue field. Let us denote by A_K a semistable Abelian variety over K, by A'_K its dual and by I_K the universal vectorial extension of A_K. So we have an exact sequence of algebraic groups over K

$$(1) \qquad 0 \to V_K \to I_K \to A_K \to 0,$$

where V_K is a vector group. Let A be the Neron model of A_K and I and V be group schemes over R (V is a vector group) with generic fibers I_K and V_K, respectively, and such that we have an exact sequence of group schemes over R

$$(2) \qquad 0 \to V \to I \to A \to 0.$$

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Now we formally complete each term of the exact sequence (2) along the identities of their special fibers (this operation will be denoted by superscript f) and get the exact sequence of formal groups over R:

$$0 \to V^f \to I^f \to A^f \to 0. \tag{3}$$

If A_K is an ordinary semistable Abelian variety, A^f is a formal group of multiplicative type, i.e., $A^f \cong (\mathbf{G}_m^f)^g$, where $g = \dim A_K$, and the isomorphism above is defined over R^{ur}, the ring of integers of K^{ur}, the completion of the maximal unramified extension of K. Therefore there exists a unique splitting of the exact sequence (3) defined over R^{ur}, which we denote by v. We denote by Lie(v) the map induced by v on the Lie algebras and by v^* the pullback by v on differentials.

As we see in section 2, the exact sequence of Lie algebras induced by (3) can be identified with the Hodge-filtration exact sequence of $H^1_{dR}(A'_K)$, i.e., we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Lie}(V^f)_K & \to & \mathrm{Lie}(I^f)_K & \to & \mathrm{Lie}(A^f)_K & \to & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \to & H^0(A'_K, \Omega^1_{A'_K}) & \to & H^1_{dR}(A'_K) & \to & \mathrm{Lie}(A_K) & \to & 0 \end{array}$$

The main result of this paper is

THEOREM 1.1: *The map* Lie(v) *is, under the identifications above, the unit root splitting of the Hodge filtration of* $H^1_{dR}(A'_K)_{K^{ur}}$.

We remark that Theorem 1.1 is known if A_K has good ordinary reduction (see [C-UVB], Proposition 3.1.2, page 641). No proof of this result is given in [C-UVB], so the present paper supplies the proof for the good reduction case as well. One might be interested in this result in relation to p-adic height pairings, namely: the method of Zarhin in [Za] allows one to produce from a splitting of the Hodge filtration of $H^1_{dR}(A'_K)$ a local p-adic height pairing. A consequence of Theorem 1.1 is that the p-adic height pairing produced by Lie(v), where v is the splitting of (3) discussed above, coincides with the p-adic height pairing produced by the unit root splitting.

The plan of the paper is the following: in section 2 we investigate properties of the Dieudonné module of a formal group $\mathcal{F}$ over R, denoted $\mathbf{D}(\mathcal{F}/R)$, and prove that if A_K is any semistable Abelian variety over K, then we have an exact sequence

$$0 \to H^1_{dR}(A_K)^{\text{slope}=0} \to H^1_{dR}(A_K) \to \mathbf{D}(A^f/K) \to 0. \tag{4}$$

This generalizes the similar result for the good reduction case in [K].

In section 3 we complete the proof of Theorem 1.1 above, analyzing the behaviour of the Poincaré (cup-product) pairing with respect to the Frobenius-equivariant decomposition of $H^1_{dR}(A_K)$ and $H^1_{dR}(A'_K)$. We prove that if A_K is ordinary semistable, the unit root subspaces of A_K and A'_K are orthogonal with respect to the Poincaré pairing. This generalizes the similar result for the good reduction case in [C-UVB].

Finally, in the Appendix we show that the Dieudonné module of a formal group $\mathcal{F}$ over R can be interpreted as the first cohomology group of a double complex attached to $\mathcal{F}$. This is not used elsewhere in the paper, but we think it is interesting and have not seen it in the literature.

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2. Dieudonné modules of formal groups à la Honda and Katz

Let us start with some notations. Let K, R, m_R and k be as in the Introduction. If X is any group scheme over R, we denote by X^f the formal completion of X along its special fiber. It is a formal group over R. If X and Y are group schemes over R and u is a pointed morphism of schemes (i.e., it preserves the identities), we denote by u^f the formal completion of u, i.e., the morphism of pointed formal schemes $X^f \to Y^f$ induced by u. If X is now any group scheme or formal group over R, we denote by $\mathrm{Lie}(X)$ and $\mathrm{Inv}(X)$ the Lie algebra and the R-module of invariant 1-forms of X, respectively. Also, if M is any group scheme over R or any R-module, we denote by M_K the object obtained from M by extending the scalars to K, i.e., $M_K\colon = M \otimes_R K$. If f is a morphism of R-group schemes or R-modules, we denote $f_K\colon = f \otimes_R \mathrm{id}_K$. Throughout the paper, whenever we have $X \subset Y$, for X, Y schemes or formal schemes, we denote by res^Y_X, or simply by res, the restriction map for differentials or de Rham cohomology. Let now A, I, A_K and I_K be as in the Introduction. Then we have

LEMMA 2.1: *$H^1_{dR}(A)_K \cong H^1_{dR}(A_K)$ and $H^1_{dR}(I)_K \cong H^1_{dR}(I_K)$, where the maps are induced by restriction to the generic fibers.*

Proof: As K is flat over R, the base change theorem for sheaf cohomology works for A and I (they are both smooth over R), and so we get isomorphisms between the Hodge to de Rham spectral sequences. Therefore, the maps induced on the graded quotients are isomorphisms and we get the conclusion.

LEMMA 2.2: *We have the following canonical identifications:*

(a) $\mathrm{Lie}(I^f)_K \cong \mathrm{Lie}(I)_K \cong \mathrm{Lie}(I_K) \cong H^1_{dR}(A'_K)$

and

(b) $\mathrm{Inv}(I^f)_K \cong \mathrm{Inv}(I)_K \cong \mathrm{Inv}(I_K) \cong H^1_{dR}(A_K)$.

Also,

(c) $\mathrm{Lie}(A^f)_K \cong \mathrm{Lie}(A)_K \cong \mathrm{Lie}(A_K) \cong H^1(A'_K, \mathcal{O}_{A'_K})$

and

(d) $\mathrm{Inv}(A^f)_K \cong \mathrm{Inv}(A)_K \cong \mathrm{Inv}(A_K) \cong H^0(A_K, \Omega^1_{A_K/K})$.

Proof: The first two isomorphisms in each row come from standard facts (see [G-D] and the proof of the Lemma 2.1). For the last isomorphism in (a) see [Ma-Me], for the last one in (b) see [C-UVB], and for the last isomorphisms in (c) and (d) see [M].

Theorem 1.1 of the Introduction mentions "the unit root splitting" of the Hodge filtration of the de Rham cohomology of an ordinary semistable Abelian variety. Let us first recall what "the unit root splitting" is. If A_K is a semistable abelian variety over K, then we have a Frobenius endomorphism on $H^1_{dR}(A_K)$, whose definition depends on a choice of a branch of the p-adic logarithm on K^*. (Actually, we have three such Frobenii, namely, the one defined by Hyodo–Kato in [H-K], one coming from Fontaine's theory of semistable Galois representations [Fo-SS], and finally one defined specifically for Abelian varieties in [C-I]. It is proved in [Ts] and [C-I] that if A_K is split semistable, then these are all the same.) So the unit root subspace of $H^1_{dR}(A_K)$ is defined to be the slope zero subspace for the action of Frobenius. It is proved in [Il], in a more general context (and a simple proof for Abelian varieties is provided in Corollary 3.1 of this paper), that this subspace is independent of the choice of Frobenius. Moreover, its intersection with the Hodge filtration is $\{0\}$. Now if A_K is ordinary, then the dimension of the unit root subspace is equal to the dimension of A_K. So the unit root splitting of the Hodge filtration of $H^1_{dR}(A_K)$ is the splitting defined by taking the unit root subspace to be the complement of $H^0(A_K, \Omega^1_{A_K/K})$. If A_K is ordinary semistable, then A'_K is also ordinary semistable.

Let us now go back to Theorem 1.1 and the notations at the beginning of the section. As the map $\mathrm{Lie}(v)_K$ can be interpreted as a splitting of the Hodge filtration of $H^1_{dR}(A'_K)$, all we need to show in order to prove the Theorem is that its image is the unit root subspace of $H^1_{dR}(A'_K)$. In order to prove this, we first prove

THEOREM 2.3: *The map v^* is, under the identifications in Lemma 2.2, the unit root splitting of the Hodge filtration of $H^1_{dR}(A_K)_{K^{ur}}$.*

For the proof of Theorem 2.3 we only need to show that $\mathrm{Ker}(v^*)$ is the unit root subspace of $H^1_{dR}(A_K)_{K^{ur}}$. Before starting the proof of Theorem 2.3 we need some results on the de Rham cohomology and Dieudonné modules of formal groups. Let $\mathcal{F}$ be an n-dimensional formal group over R (see [K] for a definition and basic properties). We denote by $A(\mathcal{F})$ the affine algebra of $\mathcal{F}$ (if we fix a system of coordinates $X = (X_1, X_2, \ldots, X_n)$ of $\mathcal{F}$, then $A(\mathcal{F}) \cong R[[X]]$). If $\mathcal{G}$ is another formal group and $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism of formal schemes (or a homomorphism of formal groups), we denote by f^* the continuous homomorphism of R-algebras (or of R-bialgebras, respectively)

$$f^*\colon A(\mathcal{G}) \to A(\mathcal{F})$$

induced by f.

If $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are formal groups over R, we say that the sequence

$$0 \to \mathcal{F} \xrightarrow{\psi} \mathcal{G} \xrightarrow{\varphi} \mathcal{H} \to 0 \tag{5}$$

is exact if it is an exact sequence of abelian sheaves on the fppf topology. In particular, the fact that the sequence is exact implies that

$$\mathcal{F} \cong \mathcal{G} \times_{\mathcal{H}, \epsilon_{\mathcal{H}}} \mathrm{Spf}(R),$$

where $\epsilon_{\mathcal{H}}\colon \mathrm{Spf}(R) \to \mathcal{H}$ is the unit section (see [Me]).

Let us denote by $\mathrm{CFG}(R)$ the additive category of commutative formal groups over R ([K]). Then if $\mathcal{F} \in \mathrm{CFG}(R)$ of dimension n, the cohomology groups $H^i_{dR}(\mathcal{F}/R)$ are the R-modules obtained by taking the cohomology of the formal de Rham complex $\Omega^{\cdot}_{\mathcal{F}/R}$ (this is the separated completion of the "literal" de Rham complex of $A(\mathcal{F})$ as R-algebra). Let $m, pr_1, pr_2\colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be the group law and the two projections (they are morphisms in $\mathrm{CFG}(R)$). Then we define the Dieudonné module of $\mathcal{F}$, $\mathbf{D}(\mathcal{F}/R)$, to be the R-submodule of $H^1_{dR}(\mathcal{F}/R)$ consisting of the primitive elements, i.e., the elements $a \in H^1_{dR}(\mathcal{F}/R)$ such that

$$m^*(a) = pr_1^*(a) + pr_2^*(a) \quad \text{in } H^1_{dR}(\mathcal{F}/R).$$

Lemma 5.1 in [K] asserts that the association: $\mathcal{F} \to \mathbf{D}(\mathcal{F}/R)$ defines a contravariant additive functor from $\mathrm{CFG}(R)$ to R-modules. We always extend scalars to K, so let us denote

$$\mathbf{D}(\mathcal{F}/K)\colon = \mathbf{D}(\mathcal{F}/R)_K\colon = \mathbf{D}(\mathcal{F}/R) \otimes_R K.$$

If $\mathcal{F}$ is a p-divisible formal group of height h (in the sense of [K]), then $\mathbf{D}(\mathcal{F}/K)$ is a vector space of dimension h. We have the following description of $\mathbf{D}(\mathcal{F}/K)$: let $T=(T_1,T_2,\ldots,T_n)$ be a system of coordinates of $\mathcal{F}$. Then we have

LEMMA 2.4: *$\mathbf{D}(\mathcal{F}/K)=\{f\in K[[T]]|\quad f(0_{\mathcal{F}})=0, df\in d(R[[X]]_K)$, and $f(X\overset{\mathcal{F}}{+}Y)-f(X)-f(Y)\in R[[X,Y]]_K\}/\{f\in R[[T]]_K|\ f(0_{\mathcal{F}})=0\}$, where $\overset{\mathcal{F}}{+}$ is the group law of $\mathcal{F}$.*

Proof: This follows easily from the description in [K] p. 193.

The next two Lemmas are obviously well-known, but we write them here for better reference.

LEMMA 2.5: *Let $\mathcal{F}=(\mathbf{G}_a^f)^n$ be defined over R and let $T=(T_1,T_2,\ldots,T_n)$ be a system of coordinates of $\mathcal{F}$. Let $F\in K[[T]]$ be a homomorphism $F\colon\mathcal{F}\to\mathbf{G}_a^f$, defined over K. Then $F\in R[[T]]_K$.*

Proof: If F is a homomorphism then $F(X+Y)=F(X)+F(Y)$, where $X=(X_1,X_2,\ldots,X_n)$ and $Y=(Y_1,Y_2,\ldots,Y_n)$ and the equality takes place in $K[[X,Y]]$. Therefore, $F(T)=a_1T_1+a_2T_2+\cdots+a_nT_n$, with $a_i\in K$, so indeed $F\in R[[T]]_K$. ∎

LEMMA 2.6: *Let $T=(T_1,T_2,\ldots,T_n)$ and $F,G\in K[[T]]$ be such that:*

(a) F,G are convergent on $(m_R)^n$ (i.e., on the R-rational points of the open unit polydisk around $(0,0,\ldots,0)$).

(b) $F(x)=G(x)$ for all $x\in(m_R)^n$.

Then $F=G$.

Proof: F,G give functions $F(R),G(R)\colon(m_R)^n\to K$, which are continuous and indefinitely differentiable at $(0,0,\ldots,0)$. The partial derivatives can be defined using limits, as in elementary calculus, and can also be computed like this: formally partial differentiate the power series and evaluate the result at $(0,0,\ldots,0)$. As $F(R)=G(R)$, all their partial derivatives at $(0,0,\ldots,0)$ are equal, so the coefficients of the power series are equal, so $F=G$. ∎

Let, as above, $T=(T_1,T_2,\ldots,T_n)$; then we define on $K[[T]]$ the linear topology τ by defining a basis of neighborhoods of zero to be

$$U_{s,t}\colon=\pi^sR[[T]]+T^tK[[T]],\quad \text{where } s,t\in\mathbf{N},$$

and where π is a uniformizer of R and we denote by $T^tK[[T]]$ the ideal $(T_1K[[T]]+T_2K[[T]]+\cdots+T_nK[[T]])^t$ in $K[[T]]$. Then we have the following

LEMMA 2.7: (a) *The topology τ is separated and its restriction to $R[[T]]$ is the usual topology on $R[[T]]$.*

(b) *If $\alpha \in \pi R[[T]] + T \cdot R[[T]]$ then we have*

$$\alpha^m/m \xrightarrow{\tau} 0 \quad \text{when } m \to \infty.$$

Proof: (a) is clear; and for (b) notice that α can be written as $\alpha = a + b$, where $a \in m_R$ and $b \in T \cdot R[[T]]$. Then, it is enough to investigate the case when $m = p^k$ and we have

$$\alpha^{p^k}/p^k \in a^{[p^k/2]}/p^k \cdot R[[T]] + T^{[p^k/2]+1}K[[T]],$$

where $[\cdot]$ denotes the greatest integer part function. So the result follows from the fact that $v(a^{[p^k/2]}/p^k) \geq [p^k/2] - ek \to \infty$ as $k \to \infty$, where e denotes the absolute ramification degree of K. ∎

Let now $\mathcal{F}: = \mathrm{Spf}(R[[T]])$ be a formal Lie variety over R and let us define $M(\mathcal{F})$ to be the R-submodule of $K[[T]]$ of all power series F such that $dF \in \Omega^1_{\mathcal{F}/R}$. Let $\mathcal{G} = \mathrm{Spf}(R[[X]])$ and $\psi: \mathcal{G} \to \mathcal{F}$ be a morphism of formal Lie varieties.

LEMMA 2.8: *There exists a unique, continuous R-module homomorphism $\Psi: M(\mathcal{F}) \to K[[X]]$ which extends ψ^*. Moreover, the image of Ψ is contained in $M(\mathcal{G})$.*

Proof: As $\psi^*: R[[T]] \to R[[X]]$ is a continuous R-algebra homomorphism we have

$$\psi^*(T_i) \in \pi \cdot R[[X]] + X \cdot R[[X]] \quad \text{for } i = 1, 2, \ldots, n.$$

So, if $F \in M(\mathcal{F})$ we define $\Psi(F) = F(\psi^*(T_1), \psi^*(T_2), \ldots, \psi^*(T_n))$. By Lemma 2.7 (b) this makes sense and we are done. ∎

Let us now consider again the exact sequence of formal groups (5):

$$0 \to \mathcal{F} \xrightarrow{\psi} \mathcal{G} \xrightarrow{\varphi} \mathcal{H} \to 0 \tag{5}$$

such that now $\mathcal{F}$ is a finite power of the formal additive group over R, i.e., $\mathcal{F} \cong (\mathbf{G}_a^f)^m$, for some $m \in \mathbf{N}$. Let us also consider a pointed morphism of formal R-schemes $u: \mathcal{H} \to \mathcal{G}$ which is a section of φ (pointed morphism means that $u(0_{\mathcal{H}}) = 0_{\mathcal{G}}$.) Then we have

$$\mathrm{Inv}(\mathcal{G})_K \xrightarrow{u_K^*} (\Omega^{1,\mathrm{cl}}_{\mathcal{H}/R})_K \xrightarrow{\mathrm{proj}} H^1_{dR}(\mathcal{H}/R)_K,$$

where $\Omega^{1,\mathrm{cl}}$ denotes the space of closed 1-forms. The following proposition is a key result.

PROPOSITION 2.9: *In the notations and hypothesis above:*

(a) *The image of* $(\mathrm{proj} \circ u_K^*)$ *is contained in* $\mathbf{D}(\mathcal{H}/K)$.

(b) *If* w *is another pointed section (as formal schemes) of* φ *then* $(\mathrm{proj} \circ u_K^*) = (\mathrm{proj} \circ w_K^*)$.

Proof: Let $\eta \in \mathrm{Inv}(\mathcal{G}/R)$ and let us denote $\omega = u^*(\eta) \in \Omega^1_{\mathcal{F}/R}$. Then as the invariant forms are closed, we have $d\omega = du^*\eta = u^*d\eta = 0$, so ω is a closed 1-form. Let $T = (T_1, T_2, \ldots, T_n)$ and $U = (U_1, U_2, \ldots, U_m)$ be systems of coordinates on $\mathcal{H}$ and $\mathcal{G}$ respectively, so $A(\mathcal{H}) \cong R[[T]]$ and $A(\mathcal{G}) \cong R[[U]]$. Let $F_\omega \in K[[T]]$ and $F_\eta \in K[[U]]$ be uniquely determined by the conditions: $dF_\omega = \omega$, $F_\omega(0) = 0$ and $dF_\eta = \eta$, $F_\eta(0) = 0$. It follows that $u^*F_\eta = F_\omega$. Let us now consider the power series: $G(X,Y) \in K[[X,Y]]$, $X = (X_1, X_2, \ldots, X_n)$, $Y = (Y_1, Y_2, \ldots, Y_n)$ defined by

$$G(X,Y) = F_\omega(X \overset{\mathcal{F}}{+} Y) - F_\omega(X) - F_\omega(Y).$$

Given the description of $\mathbf{D}(\mathcal{H}/K)$ in Lemma 2.4 we want to prove that $G(X,Y) \in R[[X,Y]]_K$. As F_ω and F_η are obtained by formal integration of closed 1-forms defined over R, they will be convergent on $\mathcal{H}(R) = (m_R)^n$ and $\mathcal{G}(R) = (m_R)^m$ respectively and they give:

$$\text{a function of sets,} \quad F_\omega(R)\colon \mathcal{H}(R) \to K,$$

and respectively

$$\text{a group homomorphism,} \quad F_\eta(R)\colon \mathcal{G}(R) \to K.$$

If X is a formal group over R, let

Δ_X:	$A(X)$	$\to$	$A(X)\hat{\otimes}A(X)$	be the comultiplication,
ϵ_X:	$A(X)$	$\to$	R	be the augmentation,
σ_X:	$A(X)$	$\to$	$A(X)$	be the coinverse,
$i_{1,X}, i_{2,X}$:	$A(X)$	$\to$	$A(X)\hat{\otimes}A(X)$	be the maps $i_{j,X} = \mathrm{pr}^*_{j,X}$, $j = 1, 2$.

Let us then define the following continuous R-algebra homomorphism:

$$f^*\colon A(\mathcal{G}) \to A(\mathcal{H})\hat{\otimes}A(\mathcal{H})$$

to be the composition:

$$A(\mathcal{G}) \overset{\Delta_\mathcal{G}}{\to} A(\mathcal{G})\hat{\otimes}A(\mathcal{G}) \overset{\Delta_\mathcal{G}\hat{\otimes}\mathrm{id}}{\to} A(\mathcal{G})\hat{\otimes}A(\mathcal{G})\hat{\otimes}A(\mathcal{G}) \overset{\mathrm{id}\hat{\otimes}\sigma_\mathcal{G}\hat{\otimes}\sigma_\mathcal{G}}{\to}$$

$$A(\mathcal{G})\hat{\otimes}A(\mathcal{G})\hat{\otimes}A(\mathcal{G}) \overset{u^*\hat{\otimes}u^*\hat{\otimes}u^*}{\to} A(\mathcal{H})\hat{\otimes}A(\mathcal{H})\hat{\otimes}A(\mathcal{H}) \overset{\Delta_\mathcal{H}\hat{\otimes}i_{1,\mathcal{H}}\hat{\otimes}i_{2,\mathcal{H}}}{\to} A(\mathcal{H})\hat{\otimes}A(\mathcal{H}).$$

Then, as we identify $R[[X,Y]] \cong A(\mathcal{H})\hat{\otimes}A(\mathcal{H})$ by $X \to i_{1,\mathcal{H}}(T) = T\hat{\otimes}1$ and $Y \to i_{2,\mathcal{H}}(T) = 1\hat{\otimes}T$, it makes sense to make the

FIRST CLAIM: (1) $G = f^*F_\eta$,
(2) $f^*\varphi^* = \epsilon^*_{\mathcal{H}}$.

Proof of the First Claim: (1) One can check the claim directly using the properties of the co-structures, but given Lemma 2.6 it would be enough to check the equality on R-points. So if we denote by f the homomorphism of formal schemes induced by f^*, we'd like to prove that $G(R)$ and $F_\eta(R)\cdot f(R)$ are equal as maps of sets from $\mathcal{H}(R)\times\mathcal{H}(R)$ to K. Let us first make the map $f(R)\colon \mathcal{H}(R)\times\mathcal{H}(R)\to\mathcal{G}(R)$ explicit. It is given by

$$f(R)(x,y) = u(R)(x \overset{\mathcal{H}}{+} y) \overset{\mathcal{G}}{-} u(R)(x) \overset{\mathcal{G}}{-} u(R)(y)$$

for all $(x,y)\in\mathcal{H}(R)\times\mathcal{H}(R)$. Therefore, as $F_\eta(R)\colon \mathcal{G}(R)\to K$ is a homomorphism we have

$$\begin{aligned} F_\eta(R)\cdot f(R)(x,y) &= F_\eta(R)(u(R)(x \overset{\mathcal{H}}{+} y) \overset{\mathcal{G}}{-} u(R)(x) \overset{\mathcal{G}}{-} u(R)(y)) \\ &= F_\omega(R)(x \overset{\mathcal{H}}{+} y) - F_\omega(R)(x) - F_\omega(R)(y) = G(R)(x,y), \end{aligned}$$

for all $(x,y)\in\mathcal{H}(R)\times\mathcal{H}(R)$.

(2) Applying the same Lemma 2.6 it is enough to check this equality on R-points, i.e., we want to show that

$$\varphi(R)\cdot f(R) = 0_{\mathcal{H}(R)}.$$

But this follows easily from the fact that $\varphi(R)$ is a group homomorphism and $\varphi(R)\cdot u(R) = \mathrm{id}_{\mathcal{H}(R)}$.

Now we can prove (a) of Proposition 2.9.

As the sequence (5) is exact we have the following cartesian diagram (i.e., $A(\mathcal{F}) \cong A(\mathcal{G})\hat{\otimes}_{A(\mathcal{H}),\epsilon_{\mathcal{H}}}R$):

$$(6)\qquad \begin{array}{ccc} A(\mathcal{H}) & \overset{\varphi^*}{\to} & A(\mathcal{G}) \\ \epsilon^*_{\mathcal{H}}\downarrow & & \downarrow\psi^* \\ R & \subset & A(\mathcal{F}) \end{array}$$

and continuous R-algebra homomorphisms $A(\mathcal{G}) \overset{f^*}{\to} A(\mathcal{H})\hat{\otimes}A(\mathcal{H})$ and $R\subset A(\mathcal{H})\hat{\otimes}A(\mathcal{H})$ such that the diagram

$$\begin{array}{ccc} A(\mathcal{H}) & \overset{\varphi^*}{\to} & A(\mathcal{G}) \\ \epsilon^*_{\mathcal{H}}\downarrow & & \downarrow f^* \\ R & \subset & A(\mathcal{H})\hat{\otimes}A(\mathcal{H}) \end{array}$$

commutes. Therefore, there exists a unique continuous R-algebra homomorphism

$$\alpha^*: A(\mathcal{F}) \to A(\mathcal{H})\hat{\otimes}A(\mathcal{H})$$

such that the diagram

$$\begin{array}{ccc} A(\mathcal{G}) & \stackrel{f^*}{\to} & A(\mathcal{H})\hat{\otimes}A(\mathcal{H}) \\ \psi^* \downarrow & & \| \\ A(\mathcal{F}) & \stackrel{\alpha^*}{\to} & A(\mathcal{H})\hat{\otimes}A(\mathcal{H}) \end{array}$$

commutes. We have $G = f^*F_\eta = \alpha^*(\psi^*F_\eta)$. As $\mathcal{F}$ is a power of the formal additive group, we deduce from Lemma 2.5 that $\psi^*F_\eta \in A(\mathcal{F})_K$; and as α^* is defined over R we get that $G \in (A(\mathcal{H})\hat{\otimes}A(\mathcal{H}))_K$.

Now we prove (b) of Proposition 2.9. Let w be a second pointed section (as formal schemes) of φ and let us denote by $\omega' := w^*(\eta)$ and by $F_{\omega'}$ the unique element of $K[[T]]$ such that $dF_{\omega'} = \omega'$ and $F_{\omega'}(0) = 0$. Then let us denote by g^* the following composition of continuous R-algebra homomorphisms:

$$A(\mathcal{G}) \stackrel{\Delta_\mathcal{G}}{\to} A(\mathcal{G})\hat{\otimes}A(\mathcal{G}) \stackrel{\mathrm{id}_{A(\mathcal{G})}\hat{\otimes}\sigma_\mathcal{G}}{\to} A(\mathcal{G})\hat{\otimes}A(\mathcal{G}) \stackrel{u^*\hat{\otimes}w^*}{\to} A(\mathcal{H})\hat{\otimes}A(\mathcal{H}) \stackrel{\text{product}}{\to} A(\mathcal{H}).$$

So, $g^*: A(\mathcal{G}) \to A(\mathcal{H})$ and we can make the

SECOND CLAIM: (1) $F_\omega - F_{\omega'} = g^*F_\eta$,
(2) $g^* \cdot \varphi^* = \epsilon_\mathcal{H}$.

The proof of the Second Claim is similar to the proof of the First Claim and is left to the reader. Moreover, (b) of Proposition 2.9 follows from the Second Claim by the same formal argument as above. ∎

Now we apply the results of Proposition 2.9 to the exact sequence (3). Let us recall it:

$$0 \to V^f \stackrel{\psi}{\to} I^f \stackrel{\varphi}{\to} A^f \to 0. \tag{3}$$

Let us also recall that we had a section of φ, which is a homomorphism and is defined over R^{ur}, denoted v. Let us construct another one (it will be defined over R and it will not be a homomorphism in general, but only a morphism of pointed formal schemes). For this, recall the exact sequence of group schemes over R:

$$0 \to V \stackrel{a}{\to} I \stackrel{b}{\to} A \to 0. \tag{2}$$

Let now $U \subset A$ be an affine open which contains the unit of the special fiber. As $\mathbf{G}_a$-torsors are locally trivial in the Zariski topology, there exists a morphism of schemes $u: U \to I$ which is a section of b. Without loss of generality we may assume that u is a pointed section (if not, compose u with the translation on I by some R-rational point of V). Now we take formal completions along identities of the special fibers and get

$$u^f : U^f \cong A^f \to I^f$$

and u^f is a pointed section of $b^f = \varphi$.

Now applying Proposition 2.9, we get that the following diagram:

$$\begin{array}{ccc} \mathrm{Inv}(I^f)_{K^{ur}} & \stackrel{v^*}{\to} & \mathrm{Inv}(A^f)_{K^{ur}} \\ \| & & \downarrow\cong \\ \mathrm{Inv}(I^f)_{K^{ur}} & \stackrel{(u^f)^*}{\to} & \mathbf{D}(A^f/K^{ur}) \end{array} \tag{7}$$

is commutative.

In the notations above we have

Proposition 2.10: $\mathrm{Ker}((u^f)^*)_K = \mathrm{Ker}(\mathrm{res}^A_{A^f}: H^1_{dR}(A)_K \to \mathbf{D}(A^f/K))$, *where, let us recall,* $\mathrm{res}^A_{A^f}$ *is the restriction map.*

Proof: We have the following commutative diagrams:

$$\begin{array}{ccc} H^1_{dR}(A)_K & \stackrel{\mathrm{res}^A_U}{\to} & H^1_{dR}(U)_K \\ \mathrm{res}^A_{A^f}\downarrow & & \downarrow\mathrm{res}^U_{U^f} \\ H^1_{dR}(A^f)_K & = & H^1_{dR}(U^f)_K \end{array}$$

and

$$\begin{array}{ccc} H^1_{dR}(A)_K & \stackrel{b^*}{\to} & H^1_{dR}(I)_K \\ \mathrm{res}^A_U\downarrow & & \downarrow u^* \\ H^1_{dR}(U)_K & = & H^1_{dR}(U)_K \end{array}$$

But we also have $\mathrm{Inv}(I_K) \cong \mathrm{Inv}(I)_K \cong H^1_{dR}(I)_K \cong H^1_{dR}(I_K)$, where all the maps are the natural ones (see Lemma 2.2 and [C-UVB]) so the map induced by b^* is an isomorphism (also [C-UVB]). Therefore, it follows that $u^* = \mathrm{res}^A_U \cdot (b^*)^{-1}$. Moreover, we have the diagram

$$\begin{array}{ccccc} \mathrm{Inv}(I)_K & \stackrel{\mathrm{res}^I_{I^f}}{\to} & \mathrm{Inv}(I^f)_K & \stackrel{(u^f)^*}{\to} & \mathbf{D}(A^f/K) \\ u^*\downarrow & & & & \cap \\ H^1_{dR}(U)_K & \stackrel{\mathrm{res}^U_{U^f}}{\to} & H^1_{dR}(U^f)_K & = & H^1_{dR}(A^f)_K \end{array}$$

This diagram is also commutative (see [EGAI], section 9). Finally, it follows that the diagram

$$\begin{array}{ccc} \mathrm{Inv}(I^f)_K & \overset{(u^f)^*}{\rightarrow} & \mathbf{D}(A^f/K) \\ \downarrow\cong & & \cap \\ H^1_{dR}(A)_K & \overset{\mathrm{res}^A_{A^f}}{\rightarrow} & H^1_{dR}(A^f)_K \end{array}$$

is commutative, which proves the Proposition. ∎

Now the proof of Theorem 2.3 will follow from a more general statement. Let now A_K be any split semistable Abelian variety. Let us first recall the "uniformization cross" of A_K: we have a diagram of rigid analytic groups and morphisms

$$(*) \qquad \begin{array}{ccccc} & & \Gamma_K & & \\ & & \downarrow & & \\ T_K & \rightarrow & G_K & \rightarrow & B_K \\ & & \downarrow & & \\ & & A_K & & \end{array}$$

where T_K is a split torus over K, B_K is an Abelian variety with good reduction over K, and Γ_K is a free Abelian group over K; T_K, G_K and B_K are algebrizable and they have canonical models over R, say T, G and B respectively. Moreover, there is an exact sequence of group schemes

$$0 \rightarrow T \rightarrow G \rightarrow B \rightarrow 0$$

over R which induces the horizontal row in the cross. We formally complete each term of this exact sequence along the identities of their special fibers and get an exact sequence of formal groups

$$0 \rightarrow T^f \rightarrow G^f \rightarrow B^f \rightarrow 0.$$

If X is any of the group schemes over R above (A, G, B, T), then the image of the map $\mathrm{res}^X_{X^f}\colon H^1_{dR}(X) \rightarrow H^1_{dR}(X^f)$ is in $\mathbf{D}(X^f/R)$. We denote by res_X the composition:

$$H^1_{dR}(X_K) \cong H^1_{dR}(X)_K \rightarrow \mathbf{D}(X^f/K).$$

LEMMA 2.11: *The map* $\mathrm{res}_T\colon H^1_{dR}(T_K) \rightarrow \mathbf{D}(T^f/K)$ *is an isomorphism.*

Proof: This is obvious, as the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Inv}(T)_K & \rightarrow & \mathrm{Inv}(T^f)_K \\ \downarrow & & \downarrow \\ H^1_{dR}(T_K) & \overset{\mathrm{res}_T}{\rightarrow} & \mathbf{D}(T^f/K) \end{array}$$

and all the other maps are isomorphisms. ∎

PROPOSITION 2.12: *Let A_K be a split semistable Abelian variety over K. Then we have an exact sequence of K-vector spaces*

$$0 \to H^1_{dR}(A_K)^{\text{slope}=0} \to H^1_{dR}(A_K) \overset{\text{res}_A}{\to} \mathbf{D}(A^f/K) \to 0,$$

where the slope is considered with respect to Frobenius.

Proof: Let us consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \text{Ker}(\text{res}_B) & \to & \text{Ker}(\text{res}_G) & \to & 0 & & \\ & & \cap & & \cap & & \| & & \\ 0 & \to & H^1_{dR}(B_K) & \to & H^1_{dR}(G_K) & \to & H^1_{dR}(T_K) & \to & 0 \\ & & \text{res}_B \downarrow & & \text{res}_G \downarrow & & \text{res}_T \downarrow & & \\ 0 & \to & \mathbf{D}(B^f/K) & \to & \mathbf{D}(G^f/K) & \to & \mathbf{D}(T^f/K) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The diagram is commutative, the vertical sequences and the middle horizontal sequence are exact and all the maps are compatible with respect to Frobenii. The fact that the bottom horizontal sequance is also exact follows from the following facts: for a p-divisible formal group $\mathcal{F}$ over R we have $\mathbf{D}(\mathcal{F}/K) \cong \mathbf{D}_{\text{classic}}(\overline{\mathcal{F}}/W(k)) \otimes_{W(k)} K$, where $\overline{\mathcal{F}}$ is the special fiber of $\mathcal{F}$ (see [K]) and the functor $\mathbf{D}_{\text{classic}}$ (the classical Dieudonné module functor of Dieudonné and Cartier) is exact. It follows that the top sequence is exact, i.e., that $\text{Ker}(\text{res}_B) \cong \text{Ker}(\text{res}_G)$. As $\text{Ker}(\text{res}_B) = H^1_{dR}(B_K)^{\text{slope}=0}$ (see [K]) and as $H^1_{dR}(T_K)^{\text{slope}=0} = 0$ it follows that $\text{Ker}(\text{res}_G) = H^1_{dR}(G_K)^{\text{slope}=0}$. Now let us consider the following diagram (see[C-I]):

$$\begin{array}{ccccccccc} 0 & \to & \text{Hom}(\Gamma, K) & \to & \text{Ker}(\text{res}_A) & \to & \text{Ker}(\text{res}_G) & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \text{Hom}(\Gamma, K) & \to & H^1_{dR}(A_K) & \to & H^1_{dR}(G_K) & \to & 0 \\ & & \downarrow & & \text{res}_A \downarrow & & \text{res}_G \downarrow & & \\ & & 0 & \to & \mathbf{D}(A^f/K) & = & \mathbf{D}(G^f/K) & \to & 0 \end{array}$$

As Frobenius acts on $\text{Hom}(\Gamma, K)$ as identity, and from the fact just established that $\text{Ker}(\text{res}_G) = H^1_{dR}(G_K)^{\text{slope}=0}$, we deduce that $\text{Ker}(\text{res}_A) = H^1_{dR}(A_K)^{\text{slope}=0}$. ∎

As a consequence of Proposition 2.10 and Proposition 2.12 we get that, if A_K is an ordinary semistable Abelian variety over K, then we have

$$\begin{aligned} \text{Ker}(v^*) = \text{Ker}((u^f)^*)_{K^{ur}} = \text{Ker}(\text{res}_A)_{K^{ur}} &= H^1_{dR}(A_K)^{\text{slope}=0}_{K^{ur}} \\ &= \text{unit root subspace,} \end{aligned}$$

which finally proves Theorem 2.3. ∎

Corollary 2.13: *As the unit root subspace of* $H^1_{dR}(A_K)$ *is* $\mathrm{Ker}(\mathrm{res}_A)$, *this subspace is absolutely canonical, although the Frobenius morphism itself depends on a choice of a branch of the p-adic logarithm.*

3. On the Poincaré pairing

Let us go back to Theorem 1.1 of the Introduction. We denote $L: = K^{ur}$ and recall that A_K was our ordinary semistable Abelian variety over K. We have maps

$$(\mathrm{Inv}(I^f))_L \xrightarrow{v^*} (\mathrm{Inv}(A^f))_L \quad \text{and} \quad (\mathrm{Lie}(A^f))_L \xrightarrow{\mathrm{Lie}(v)} (\mathrm{Lie}(I^f))_L,$$

and also, we have perfect pairings

$$<,> : (\mathrm{Inv}(I^f))_L \times (\mathrm{Lie}(I^f))_L \to L \quad \text{and} \quad <,> : (\mathrm{Inv}(A^f))_L \times (\mathrm{Lie}(A^f))_L \to L.$$

If $x \in (\mathrm{Inv}(I^f))_L$ and $y \in (\mathrm{Lie}(A^f))_L$, we have

$$< v^*(x), y >=< x, \mathrm{Lie}(v)(y) >,$$

therefore we deduce that under the pairings above, the subspaces $\mathrm{Ker}(v^*)_L$ and $\mathrm{Im}(\mathrm{Lie}(v))_L$ are orthogonal. As we base change from K to L, it is harmless to assume that our Abelian variety A_K is actually split semistable over K, which we do. Hence in order to prove Theorem 1.1 we only need

Theorem 3.1: *Suppose* A_K *is a split semistable Abelian variety. Then under the Poincaré pairing*

$$<,>_{\mathrm{Poin},A} : H^1_{dR}(A_K) \times H^1_{dR}(A'_K) \to K$$

the unit root subspaces are orthogonal.

Remark 3.1: The statement of the Theorem is well-known if A_K has good ordinary reduction; see [C-UVB].

Remark 3.2: We first remark that the Poincaré pairing is the same as the pairing obtained from the identifications in Lemma 2.2 and the pairing above (see [C-DA]).

Remark 3.3: Theorem 3.1 is actually a consequence of a more general result, namely: if A_K is a split semistable Abelian variety then we have

$$< \Phi_{A_K}(a), \Phi_{A'_K}(b) >_{\text{Poin}} = q < a, b >_{\text{Poin}},$$

where Φ_{A_K} and $\Phi_{A'_K}$ are K-linear Frobenii on $H^1_{dR}(A_K)$ and $H^1_{dR}(A'_K)$ respectively and $a \in H^1_{dR}(A_K)$, $b \in H^1_{dR}(A'_K)$. This formula is known to hold if A_K has good reduction, so Theorem 3.1 is true in this case. The proof of the above formula for split semistable A_K will be supplied in [CI-FMD] and is surprisingly difficult.

In order to prove Theorem 3.1 we need to study the behaviour of the Poincaré pairing with respect to Frobenius. Let us recall the uniformization cross $(*)$ of the semistable Abelian variety A_K in section 2 and that we have fixed a branch of the p-adic logarithm on K^*. So we have exact sequences of filtered, Frobenius, monodromy modules:

$$0 \to \text{Hom}(\Gamma, K) \to H^1_{dR}(A_K) \to H^1_{dR}(G_K) \to 0$$

and

$$0 \to H^1_{dR}(B_K) \to H^1_{dR}(G_K) \to H^1_{dR}(T_K) \to 0.$$

Moreover, these exact sequences are naturally Frobenius-equivariant split (see [C-I]), so there exist K-vector subspaces of $H^1_{dR}(A_K)$ invariant under Frobenius: $U(A), V(A), W(A)$, such that

(a) $H^1_{dR}(A_K) = U(A) \oplus V(A) \oplus W(A)$

and

(b) $U(A) \cong \text{Hom}(\Gamma, K)$, $V(A) \cong H^1_{dR}(B_K)$ and $W(A) \cong H^1_{dR}(T_K)$, where these K-linear isomorphisms are Frobenius-equivariant.

We first prove

PROPOSITION 3.2: *Let $a = (x, y) \in U(A) \oplus V(A) \subset H^1_{dR}(A_K)$ and $a' = (x', y') \in U(A') \oplus V(A') \subset H^1_{dR}(A'_K)$. Then, under the identifications in* (b) *above we have*

$$< a, a' >_{\text{Poin},A} = < y, y' >_{\text{Poin},B} .$$

Proof: Let us denote by h_A the composition of the K-linear maps $H^1_{dR}(A_K) \overset{f_A}{\to} H^1_{dR}(G_K) \overset{g_A}{\to} H^1_{dR}(T_K)$, where f_A and g_A are the natural maps. The Proposition will follow from the following sequence of results.

LEMMA 3.3: *Under the pairing* $<,>_{\mathrm{Poin},A}$, *we have*

$$U(A)^{\perp}=\mathrm{Ker}(h_{A'})=U(A')\oplus V(A').$$

Proof: If we denote by N_A: $H^1_{dR}(A_K)\to U(A)$ the monodromy operator, it is proved in [C-I] that N_A is surjective. Moreover, Theorem 3.1 of chapter I of [C-I] gives, for $b\in U(A)$, $b'\in H^1_{dR}(A'_K)$ and $x\in H^1_{dR}(A_K)$ such that $N_A(x)=b$:

$$<b,b'>_{\mathrm{Poin},A}=<N_A(x),b'>_{\mathrm{Poin},A}=<h_A(x),h_{A'}(b')>_{\mathrm{mon}},$$

where $<,>_{\mathrm{mon}}$ is the monodromy pairing. Therefore $\mathrm{Ker}(h_{A'})=U(A')\oplus V(A')\subset U(A)^{\perp}$. A dimension count now finishes the proof. ∎

PROPOSITION 3.4 (R. Coleman): *Let* $a\in\mathrm{Ker}(h_A)$ *and* $a'\in\mathrm{Ker}(h_{A'})$. *Then we have*

$$<a,a'>_{\mathrm{Poin},A}=<f_A(a),f_{A'}(a')>_{\mathrm{Poin},B}.$$

Proof: It would be enough to prove the lemma if $A=A'$ is the Jacobian of a semistable curve X. Let us recall some notations and results from chapter I of [C-I]. Let X be a smooth, connected complete curve over K, with a regular semistable model $\mathcal{X}$ over R such that the irreducible components of its reduction $\overline{\mathcal{X}}$ are smooth; suppose that there are at least two of them, and that they, as well as the singular points of $\overline{\mathcal{X}}$, are defined over k.

Let $\mathrm{Gr}(\mathcal{X})$ be the graph with oriented edges defined as follows: the vertices $V(\mathcal{X})$ of $\mathrm{Gr}(\mathcal{X})$ are the irreducible components of $\overline{\mathcal{X}}$. Let $\overline{\mathcal{X}}^n$ denote the normalization of $\overline{\mathcal{X}}$ and n: $\overline{\mathcal{X}}^n\to\overline{\mathcal{X}}$ be the natural map. The edges $E(\mathcal{X})$ of $\mathrm{Gr}(\mathcal{X})$ will be symbols $[x,y]$, where x,y are points of $\overline{\mathcal{X}}^n(\overline{k})$, whose images under n, in $\overline{\mathcal{X}}(\overline{k})$, are the same. We set $A([x,y])$ equal to the image in $\overline{\mathcal{X}}$ of the component of $\overline{\mathcal{X}}^n$ on which x lies and $B([x,y])$ the image in $\overline{\mathcal{X}}$ of the component of $\overline{\mathcal{X}}^n$ on which y lies. Then if $e=[x,y]\in E(\mathcal{X})$, e will be an edge from $A(e)$ to $B(e)$. If Y is a subscheme of $\overline{\mathcal{X}}$, we denote by X_Y the tube of Y, considered as a rigid subspace of X. We have a natural involution τ of $E(\mathcal{X})$, given by $\tau([x,y])=[y,x]$. If $e\in E(\mathcal{X})$, we set $X_e:=X_{n(e)}$. We remark that $\mathcal{C}:=\{X_A \mid A\in V(\mathcal{X})\}$ is an admissible cover of X by basic wide opens. Let

$$X^0:=\coprod_{A\in V(\mathcal{X})}X_A \quad\text{and}\quad X^1:=\coprod_{e\in E(\mathcal{X})}X_e.$$

Let i be the involution on X^1 which takes a point in $X_e\subset X^1$ to the corresponding point in $X_{\tau(e)}$. We have the Meyer–Vietoris exact sequence

$$\to H^0_{dR}(X^0)\overset{a}{\to}H^0_{dR}(X^1)^-\overset{\partial}{\to}H^1_{dR}(X)\overset{\alpha}{\to}H^1_{dR}(X^0)\overset{b}{\to}H^1_{dR}(X^1)^-\to$$

where the superscript "-" denotes the -1-eigenspace for the action of i and the map a is defined as follows: Let $(x_v)_v \in H^0_{dR}(X^0)$ and $e \in E(\mathcal{X})$. Then $(a(x_v)_v)_e = x_{o(e)} - x_{t(e)}$, where $o(e)$ denotes the origin of the oriented edge e and $t(e)$ denotes the target of e. Similarly, the map b is defined by: Let $(y_v))v \in H^1_{dR}(X^0)$ and $e \in E(\mathcal{X})$. Then $(b(y_v)_v)_e = y_{o(e)}|_{X_e} - y_{t(e)}|_{X_e}$.

Let A_K be the Jacobian of X. It is a semistable Abelian variety over K, and suppose it has the uniformization cross $(*)$ in section 2. Then as shown in chapter I of [C-I] we have canonical isomorphisms

$$H^0_{dR}(X^1)^-/\operatorname{Im}(a) \cong \operatorname{Hom}(\Gamma, K) \quad \text{and} \quad \operatorname{Im}(\alpha) \cong H^1_{dR}(G_K).$$

We denote by

$$H^1_{par}(X^0) := \operatorname{Ker}(H^1_{dR}(X^0) \overset{\text{res}}{\to} H^1_{dR}(X^1)) = \operatorname{Ker}(H^1_{dR}(X^0) \overset{\text{Res}}{\to} H^0_{dR}(X^1)),$$

where, let us recall, res is the restriction map and Res is the residue along the annuli in X^1.

We now start the proof of Proposition 3.4. Let $a, a' \in \operatorname{Ker} h_A \subset H^1_{dR}(A_K)$ and $x, x' \in H^1_{dR}(X)$ be the restrictions of a, a' respectively to X. Then $\operatorname{res}^X_{X^0}(x), \operatorname{res}^X_{X^0}(x') \in H^1_{\text{par}}(X^0)$ and we have

$$< a, a' >_{\text{Poin},A} = < x, x' >_{\text{Poin},X} = < \operatorname{res}^X_{X^0}(x), \operatorname{res}^X_{X^0}(x') >_{\text{Poin},X^0},$$

where the last equality follows from Lemma 3.3. Now we need to relate this to the Poincaré pairing on B_K. By adding disks to X^0 we can view X^0 as the complement of a disjoint union of closed disks in a (non-connected) proper curve Z with good reduction. We have

$$H^1_{\text{par}}(X^0) \cong H^1_{dR}(Z),$$

and this isomorphism respects cup-products. Let D be the Jacobian of Z. Then D and B have canonically isomorphic reductions, so canonically isomorphic first crystalline cohomology groups. Therefore

$$H^1_{\text{par}}(X^0) \cong H^1_{dR}(Z) \cong H^1_{dR}(D_K) \cong H^1_{dR}(B_K),$$

and the isomorphisms above respect the cup-products. This proves Proposition 3.4 and Proposition 3.2. ∎

Remark 3.4: The referee pointed out to us that a direct proof of Proposition 3.4 (direct in the sense that it does not use reduction to Jacobians) can be found in [LS] (Proposition 6.2, part 3). As this paper has not been published we prefer to retain the proof above.

Theorem 3.1 follows from the fact that

$$H^1_{dR}(A_K)^{\text{slope}=0} = U(A) \oplus V(A)^{\text{slope}=0}$$

and

$$H^1_{dR}(A'_K)^{\text{slope}=0} = U(A') \oplus V(A')^{\text{slope}=0}$$

and that Theorem 3.1 is true for B_K, B'_K (see Remark 3.3).

4. Appendix

Let us consider a formal group $\mathcal{F}$, of dimension n over R (notations as in section 2). Let m, pr_1, pr_2: $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be the group law and the two projections respectively. Let us consider the double complex of R-modules

$$\mathcal{C}^{\bullet\bullet}: \quad \begin{array}{ccccccc} A(\mathcal{F})_0 & \stackrel{d}{\to} & \Omega^1_{\mathcal{F}/R} & \stackrel{d}{\to} & \Omega^2_{\mathcal{F}/R} & \stackrel{d}{\to} & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ A(\mathcal{F}\times\mathcal{F})_0 & \stackrel{d}{\to} & \Omega^1_{\mathcal{F}\times\mathcal{F}/R} & \stackrel{d}{\to} & \Omega^2_{\mathcal{F}\times\mathcal{F}/R} & \stackrel{d}{\to} & \cdots \end{array}$$

where $\delta = m^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*$ and $A(\mathcal{F})_0$ and $A(\mathcal{F}\times\mathcal{F})_0$ denote the sub-R-modules of $A(\mathcal{F})$ and respectively $A(\mathcal{F}\times\mathcal{F})$, consisting of power series with constant term zero. We have

Proposition 4.1: *$H^0(\mathcal{C}^{\bullet\bullet}) = 0$ and $H^1(\mathcal{C}^{\bullet\bullet}) \cong \mathbf{D}(\mathcal{F}/R)$.*

Proof: The simple complex of R-modules attached to $\mathcal{C}^{\bullet\bullet}$ is

$$K^\bullet: \quad A(\mathcal{F})_0 \stackrel{D_0}{\to} A(\mathcal{F}\times\mathcal{F})_0 \oplus \Omega^1_{\mathcal{F}/R} \stackrel{D_1}{\to} \Omega^1_{\mathcal{F}\times\mathcal{F}/R} \oplus \Omega^2_{\mathcal{F}/R} \stackrel{D_2}{\to} \cdots$$

where $D_0(F){:} = (\delta(F), d(F))$, $D_1(G,\omega){:} = (d(G) - \delta(\omega), d(\omega))$, etc., for $F \in A(\mathcal{F})_0$ and $(G,\omega) \in A(\mathcal{F}\times\mathcal{F})_0 \oplus \Omega^1_{\mathcal{F}/R}$.

One can clearly see that $H^0(\mathcal{C}^{\bullet\bullet}) = H^0(K^\bullet) = \mathrm{Ker}(D_0) = 0$ and that the map

$$H^1(\mathcal{C}^{\bullet\bullet}) = H^1(K^\bullet) = \mathrm{Ker}(D_1)/\mathrm{Im}(D_0) \to \mathbf{D}(\mathcal{F}/R)$$

given by $\mathrm{class}(G,\omega) \to \mathrm{class}(\omega)$ is an isomorphism. ∎

Corollary 4.2: *The association $\mathcal{F} \to \mathbf{D}(\mathcal{F})$ is a contravariant additive functor.*

The proof follows immediatly from Proposition 4.1. ∎

Remark 4.1: The statement of Corollary 4.2 is well-known but the approach in [K], for example, is somehow ad hoc.

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